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13. ABSTRACT *(Maximum 200 words)*
Methods for representing multi-dimensional objects, such as functions of several variables and, more generally, (hyper-)surfaces is the main objective. One goal of such representation, whether approximate or exact, is the efficient evaluation of the object: *Multivariate Polynomial Interpolation* as well as *Scattered Data Approximation* both fall into this category. Another goal is a representation that allows one to identify and access easily and simultaneously relevant aspects of the object. The topics of *wavelets* and *Weyl-Heisenberg systems* belong to this second category.

As to multivariate polynomial interpolation, we hope ultimately to duplicate the success of univariate polynomial interpolation as a basic tool in scientific computing. Current focus is the derivation of error formulae, using a multivariate divided difference just developed.

Scattered data approximation is to be accomplished by extending well-known and efficient techniques for fitting to data on uniform meshes.

Concerning Wavelets and Weyl-Heisenberg systems, the goal is to make new inroads in these important areas by applying tools and techniques from Approximation Theory, particularly those developed during studies of Shift-Invariant Spaces, a centerpiece of our previous research.

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Multivariate Approximation

Final Report

Carl de Boor & Amos Ron
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for the
U. S. Army Research Office
DAAH04-95-1-0089

Center for the Mathematical Sciences
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A. STATEMENT OF THE PROBLEM STUDIED

Methods for representing multi-dimensional objects, such as functions of several variables and, more generally, (hyper-)surfaces is the main objective. One goal of such representation, whether approximate or exact, is the efficient evaluation of the object: *Multivariate Polynomial Interpolation* as well as *Scattered Data Approximation* both fall into this category. Another goal is a representation that allows one to identify and access easily and simultaneously relevant aspects of the object. The topics of *wavelets* and *Weyl-Heisenberg systems* belong to this second category.

As to multivariate polynomial interpolation, we hope ultimately to duplicate the success of univariate polynomial interpolation as a basic tool in scientific computing. Current focus is the derivation of error formulæ, using a multivariate divided difference just developed.

Scattered data approximation is to be accomplished by extending well-known and efficient techniques for fitting to data on uniform meshes.

Concerning Wavelets and Weyl-Heisenberg systems, the goal is to make new inroads in these important areas by applying tools and techniques from Approximation Theory, particularly those developed during studies of Shift-Invariant Spaces, a centerpiece of our previous research.

B. SUMMARY OF THE MOST IMPORTANT RESULTS

This brief outline relies on the fact that more details can be found in the (semi-)annual reports submitted during the grant period, and, if need be, in the manuscripts filed with ARO as required. All bracketed numbers refer to items in the list of publications given in **C** and in the Bibliography.

A new theory for *wavelet frames* was developed in [13]. The theory is based on tools and observations that were developed in the context of *shift-invariant spaces*, and in particular the 'fiberization' of the basic operators associated with a shift-invariant system, [20], [22] (as part of work on an earlier grant). The new theory of wavelet frames is based on the interplay between the given wavelet system, and a new, closely related, shift-invariant system: *a quasi-wavelet system*. The theory led to new algorithms for constructing wavelet frames from Multiresolution Analysis. The highlight of these algorithms is that there is essentially no restriction in them on the scaling function. As a result, compactly supported spline tight frames were constructed (in one and in several variables) [16]. The new frames may be useful for various wavelet applications, most notably in the areas of feature detection and noise removal. A study of the application of the aforementioned theory was initiated during the duration of this grant.

Several other studies in the interface between Approximation Theory and Wavelets were successfully completed. Of note is the theory of *approximation orders of finitely generated shift-invariant (FSI) spaces* (cf. [5]). The theory unravels some of the previous 'mysteries' in the studies of wavelets (such as the so-called 'sum-rules'), and was applied in the search, [10], of new factorization techniques of wavelet masks in one dimension. The connection between the approximation orders of wavelets and their smoothness was also studied, [11].

A better understanding of the error formula in the proposers' least polynomial interpolant was reached for special cases (see [4]), and a `matlab` code for the construction and evaluation of the least interpolant was extended to arbitrary dimensions and made available on the web (see `http://www.cs.wisc.edu/`~**deboor/multiint/m_files.html**).

Substantial progress has been made in the area of *approximation to scattered data.* First, a new method was developed, [21], that allows the conversion of the (readily available, and very well-understood) approximation schemes on uniform grids to schemes suitable for scattered data approximation. A concrete algorithm based on the general conversion method was developed in the Ph.D. thesis of **Jungho Yoon**. The algorithm deals with approximation to scattered data on bounded planar domains. Numerical experiments with a `Matlab` code developed for that purpose showed that the algorithm performs at least as well as, and in many instances better than, the state-of-the-art methods for scattered data approximation.

Some progress was made on the problem of proper parametrization of curves for use in the construction of surfaces by blending, in that **Scott Kersey** completed a thesis on the construction of a smoothest curve that fits, within given tolerances, to a given point sequence in any number of dimensions, with the smoothness measure being minimized also with respect to the parametrization used.

C. LIST OF ALL PUBLICATIONS AND TECHNICAL REPORTS

[1] C. de Boor, *On the Sauer-Xu formula for the error in multivariate polynomial interpolation*, Math. Comp., 65 (1996), pp. 1231–1234.

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[5] C. de Boor, R. DeVore, and A. Ron, *Approximation orders of FSI spaces in* $L_2(\mathbb{R}^d)$, Constr. Approx., 14 (1998), pp. 631–652.

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[13] A. Ron and Zuowei Shen, *Affine systems in $L_2(\mathbb{R}^d)$: the analysis of the analysis operator*, J. Funct. Anal., 148 (1997), pp. 408–447.

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[15] A. Ron and Zuowei Shen, *Weyl-Heisenberg frames and Riesz bases in $L_2(\mathbb{R}^d)$*, Duke Math. J., 89 (1997), pp. 237–282.

[16] A. Ron and Zuowei Shen, *Compactly supported tight affine spline frames in $L_2(\mathbb{R}^d)$*, Math. Comp., 67 (1998), pp. 191–207.

[17] A. Ron and Zuowei Shen, *Construction of compactly supported affine frames in $L_2(\mathbb{R}^d)$*, Advances in Wavelets (K. S. Lau, ed), Springer-Verlag, New York, 1998, pp. xxx–xxx.

[18] A. Ron and Zuowei Zhen, *The Sobolev regularity of refinable functions*, submitted, 1997.

[19] A. Ron and Xingping Sun, *Strictly positive definite functions on spheres*, Math. Comp., 65(216) (1996), pp. 1513–1530.

D. LIST OF ALL PARTICIPATING SCIENTIFIC PERSONNEL SHOWING ANY ADVANCED DEGREES EARNED BY THEM WHILE EMPLOYED ON THE PROJECT

C. de Boor (entire time), A. Ron (entire time).

Two students were supported occasionally. Scott Kersey will officially finish in Spring'99, but has already successfully defended his thesis on "A minimizing spline curve under near-interpolatory constraints'. Jungho Yoon finished in Spring'98, with a thesis on "Approximation in $L_p(\mathbb{R}^d)$ from a space spanned by the scattered shifts of radial basis function".

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